Consistency

In classical, deductive logic, a **consistent** theory is one that does not lead to a logical contradiction.[1] A theory T is consistent if there is no formula φ such that both φ and its negation $\neg\varphi$ are elements of the set of consequences of T. Let A be a set of closed sentences (informally "axioms") and $\langle A \rangle$ the set of closed sentences provable from A under some (specified, possibly implicitly) formal deductive system. The set of axioms A is **consistent** when there is no formula φ such that $\varphi \in \langle A \rangle$ and $\neg\varphi \in \langle A \rangle$. A *trivial* theory (i.e., one which proves every sentence in the language of the theory) is clearly inconsistent. Conversely, in an explosive formal system (e.g., classical or intuitionistic propositional or first-order logics) every inconsistent theory is trivial.[2]:7 Consistency of a theory is a syntactic notion, whose semantic counterpart is satisfiability. A theory is satisfiable if it has a model, i.e., there exists an interpretation under which all axioms in the theory are true.[3] This is what *consistent* meant in traditional Aristotelian logic, although in contemporary mathematical logic the term *satisfiable* is used instead.

In a sound formal system, every satisfiable theory is consistent, but the converse does not hold. If there exists a deductive system for which these semantic and syntactic definitions are equivalent for any theory formulated in a particular deductive logic, the logic is called **complete**. The completeness of the propositional calculus was proved by Paul Bernays in 1918[4] and Emil Post in 1921,[5] while the completeness of (first order) predicate calculus was proved by Kurt Gödel in 1930,[6] and consistency proofs for arithmetics restricted with respect to the induction axiom schema were proved by Ackermann (1924), von Neumann (1927) and Herbrand (1931).[7] Stronger logics, such as second-order logic, are not complete.

A **consistency proof** is a mathematical proof that a particular theory is consistent.[8] The early development of mathematical proof theory was driven by the desire to provide finitary consistency proofs for all of mathematics as part of Hilbert's program. Hilbert's program was strongly impacted by the incompleteness theorems, which showed that sufficiently strong proof theories cannot prove their consistency (provided that they are consistent).

Although consistency can be proved using model theory, it is often done in a purely syntactical way, without any need to reference some model of the logic. The cut-elimination (or equivalently the normalization of the underlying calculus if there is one) implies the consistency of the calculus: since there is no cut-free proof of falsity, there is no contradiction in general.

Consistency and completeness in arithmetic and set theory

In theories of arithmetic, such as Peano arithmetic, there is an intricate relationship between the consistency of the theory and its completeness. A theory is complete if, for every formula φ in its language, at least one of φ or ¬φ is a logical consequence of the theory.

Presburger arithmetic is an axiom system for the natural numbers under addition. It is both consistent and complete.

Gödel's incompleteness theorems show that any sufficiently strong recursively enumerable theory of arithmetic cannot be both complete and consistent. Gödel's theorem applies to the theories of Peano arithmetic (PA) and primitive recursive arithmetic (PRA), but not to Presburger arithmetic.

Moreover, Gödel's second incompleteness theorem shows that the consistency of sufficiently strong recursively enumerable theories of arithmetic can be tested in a particular way. Such a theory is consistent if and only if it does *not* prove a particular sentence, called the Gödel sentence of the theory, which is a formalized statement of the claim that the theory is indeed consistent. Thus the consistency of a sufficiently strong, recursively enumerable, consistent theory of arithmetic can never be proven in that system itself. The same result is true for recursively enumerable theories that can describe a strong enough fragment of arithmetic—including set theories such as Zermelo–Fraenkel set theory (ZF). These set theories cannot prove their own Gödel sentence—provided that they are consistent, which is generally believed.

Because consistency of ZF is not provable in ZF, the weaker notion **relative consistency** is interesting in set theory (and in other sufficiently expressive axiomatic systems). If T is a theory and A is an additional axiom, $T + A$ is said to be consistent relative to T (or simply that A is consistent with T) if it can be proved that if T is consistent then $T + A$ is consistent. If both A and $\neg A$ are consistent with T, then A is said to be independent of T.

First-order logic

Notation

In the following context of mathematical logic, the turnstile symbol $\vdash$ means "provable from". That is, $a \vdash b$ reads: b is provable from a (in some specified formal system).

Definition

- A set of formulas Φ in first-order logic is **consistent** (written $\operatorname{Con}\Phi$) if there is no formula φ such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. Otherwise Φ is **inconsistent** (written $\operatorname{Inc}\Phi$).
- Φ is said to be **simply consistent** if for no formula φ of Φ, both φ and the negation of φ are theorems of Φ.
- Φ is said to be **absolutely consistent** or **Post consistent** if at least one formula in the language of Φ is not a theorem of Φ.
- Φ is said to be **maximally consistent** if Φ is consistent and for every formula φ, $\operatorname{Con}(\Phi \cup \{\varphi\})$ implies $\varphi \in \Phi$.
- Φ is said to **contain witnesses** if for every formula of the form $\exists x\, \varphi$ there exists a term t such that $(\exists x\, \varphi \to \varphi\frac{t}{x}) \in \Phi$, where $\varphi\frac{t}{x}$ denotes the substitution of each x in φ by a t; see also First-order logic.

Basic results

1. The following are equivalent:

   1. $\mathbf{Inc}\,\Phi$
   2. For all φ, $\Phi \vdash \varphi$.
2. Every satisfiable set of formulas is consistent, where a set of formulas Φ is satisfiable if and only if there exists a model $\mathfrak{I}$ such that $\mathfrak{I} \models \Phi$.
3. For all Φ and φ:
   1. if not $\Phi \vdash \varphi$, then $\mathbf{Con}(\Phi \cup \{\neg\varphi\})$;
   2. if $\mathbf{Con}\,\Phi$ and $\Phi \vdash \varphi$, then $\mathbf{Con}(\Phi \cup \{\varphi\})$;
   3. if $\mathbf{Con}\,\Phi$, then $\mathbf{Con}(\Phi \cup \{\varphi\})$ or $\mathbf{Con}(\Phi \cup \{\neg\varphi\})$.
4. Let Φ be a maximally consistent set of formulas and suppose it contains witnesses. For all φ and ψ:
   1. if $\Phi \vdash \varphi$, then $\varphi \in \Phi$,
   2. either $\varphi \in \Phi$ or $\neg\varphi \in \Phi$,
   3. $(\varphi \lor \psi) \in \Phi$ if and only if $\varphi \in \Phi$ or $\psi \in \Phi$,
   4. if $(\varphi \to \psi) \in \Phi$ and $\varphi \in \Phi$, then $\psi \in \Phi$,
   5. $\exists x\,\varphi \in \Phi$ if and only if there is a term t such that $\varphi\frac{t}{x} \in \Phi$.

Henkin's theorem

Let S be a set of symbols. Let Φ be a maximally consistent set of S-formulas containing witnesses.

Define an equivalence relation $\sim$ on the set of S-terms by $t_0 \sim t_1$ if $t_0 \equiv t_1 \in \Phi$, where $\equiv$ denotes equality. Let $\overline{t}$ denote the equivalence class of terms containing t; and let $T_\Phi := \{\, \overline{t} \mid t \in T^S \}$ where T^S is the set of terms based on the set of symbols S.

Define the S-structure $\mathfrak{T}_\Phi$ over T_Φ, also called the **term-structure** corresponding to Φ, by:

1. for each n-ary relation symbol $R \in S$, define $R^{\mathfrak{T}_\Phi}\overline{t_0}\ldots\overline{t_{n-1}}$ if $Rt_0\ldots t_{n-1} \in \Phi$;[9]
2. for each n-ary function symbol $f \in S$, define $f^{\mathfrak{T}_\Phi}(\overline{t_0}\ldots\overline{t_{n-1}}) := \overline{ft_0\ldots t_{n-1}}$;
3. for each constant symbol $c \in S$, define $c^{\mathfrak{T}_\Phi} := \overline{c}$.

Define a variable assignment β_Φ by $\beta_\Phi(x) := \bar{x}$ for each variable x. Let $\mathfrak{I}_\Phi := (\mathfrak{T}_\Phi, \beta_\Phi)$ be the **term interpretation** associated with Φ.

Then for each S-formula φ:

$$\mathfrak{I}_\Phi \models \varphi \text{ if and only if } \varphi \in \Phi.$$

Sketch of proof

There are several things to verify. First, that $\sim$ is in fact an equivalence relation. Then, it needs to be verified that (1), (2), and (3) are well defined. This falls out of the fact that $\sim$ is an equivalence relation and also requires a proof that (1) and (2) are independent of the choice of $t_0, \ldots, t_{n-1}$ class representatives. Finally, $\mathfrak{I}_\Phi \models \varphi$ can be verified by induction on formulas.

Model theory

In ZFC set theory with classical first-order logic,[10] an **inconsistent** theory T is one such that there exists a closed sentence φ such that T contains both φ and its negation φ'. A **consistent** theory is one such that the following logically equivalent conditions hold

1. $\{\varphi, \varphi'\} \not\subseteq T$[11]
2. $\varphi' \notin T \lor \varphi \notin T$

See also

Philosophy portal

- Cognitive dissonance
- Equiconsistency
- Hilbert's problems
- Hilbert's second problem
- Jan Łukasiewicz
- Paraconsistent logic
- ω-consistency
- Gentzen's consistency proof
- Proof by contradiction

Notes

1. Tarski 1946 states it this way: "A deductive theory is called *consistent* or *non-contradictory* if no two asserted statements of this theory contradict each other, or in other words, if of any two contradictory sentences … at least one cannot be proved," (p. 135) where Tarski defines *contradictory* as follows: "With the help of the word *not* one forms the *negation* of any sentence; two sentences, of which the first is a negation of the second, are called *contradictory sentences*" (p. 20). This definition requires a notion of "proof". Gödel 1931 defines the notion this way: "The class of *provable formulas* is defined to be the smallest class of formulas that contains the axioms and is closed under the relation "immediate consequence", i.e., formula *c* of *a* and *b* is defined as an *immediate consequence* in terms of *modus ponens* or substitution; cf Gödel 1931, van Heijenoort 1967, p. 601. Tarski defines "proof" informally as "statements follow one another in a definite order according to certain principles … and accompanied by considerations intended to establish their validity [true conclusion] for all true premises – Reichenbach 1947, p. 68]" cf Tarski 1946, p. 3. Kleene 1952 defines the notion with respect to either an induction or as to paraphrase) a finite sequence of formulas such that each formula in the sequence is either an axiom or an "immediate consequence" of the preceding formulas; "A *proof is said to be a proof* of *its last*

formula, and this formula is said to be (formally) provable *or be a* (formal) theorem" cf Kleene 1952, p. 83.

2. Carnielli, Walter; Coniglio, Marcelo Esteban (2016). *Paraconsistent logic: consistency, contradiction and negation*. Logic, Epistemology, and the Unity of Science. Vol. 40. Cham: Springer. doi:10.1007/978-3-319-33205-5 (https://doi.org/10.1007%2F978-3-319-33205-5). ISBN 978-3-319-33203-1. MR 3822731 (https://mathscinet.ams.org/mathscinet-getitem?mr=3822731). Zbl 1355.03001 (https://zbmath.org/?format=complete&q=an:1355.03001).
3. Hodges, Wilfrid (1997). *A Shorter Model Theory*. New York: Cambridge University Press. p. 37. "Let L be a signature, T a theory in $L_{\infty\omega}$ and φ a sentence in $L_{\infty\omega}$. We say that φ is a *consequence* of T, or that T *entails* φ, in symbols $T \vdash \varphi$, if every model of T is a model of φ. (In particular if T has no models then T entails φ.)
   Warning: we don't require that if $T \vdash \varphi$ then there is a proof of φ from T. In any case, with infinitary languages, it's not always clear what would constitute proof. Some writers use $T \vdash \varphi$ to mean that φ is deducible from T in some particular formal proof calculus, and they write $T \models \varphi$ for our notion of entailment (a notation which clashes with our $A \models \varphi$). For first-order logic, the two kinds of entailment coincide by the completeness theorem for the proof calculus in question.
   We say that φ is *valid*, or is a *logical theorem*, in symbols $\vdash \varphi$, if φ is true in every L-structure. We say that φ is *consistent* if φ is true in some L-structure. Likewise, we say that a theory T is *consistent* if it has a model.
   We say that two theories S and T in L infinity omega are equivalent if they have the same models, i.e. if Mod(S) = Mod(T)." (Please note the definition of Mod(T) on p. 30 ...)
4. van Heijenoort 1967, p. 265 states that Bernays determined the *independence* of the axioms of *Principia Mathematica*, a result not published until 1926, but he says nothing about Bernays proving their *consistency*.
5. Post proves both consistency and completeness of the propositional calculus of PM, cf van Heijenoort's commentary and Post's 1931 *Introduction to a general theory of elementary propositions* in van Heijenoort 1967, pp. 264ff. Also Tarski 1946, pp. 134ff.
6. cf van Heijenoort's commentary and Gödel's 1930 *The completeness of the axioms of the functional calculus of logic* in van Heijenoort 1967, pp. 582ff.
7. cf van Heijenoort's commentary and Herbrand's 1930 *On the consistency of arithmetic* in van Heijenoort 1967, pp. 618ff.
8. A consistency proof often assumes the consistency of another theory. In most cases, this other theory is Zermelo–Fraenkel set theory with or without the axiom of choice (this is equivalent since these two theories have been proved equiconsistent; that is, if one is consistent, the same is true for the other).
9. This definition is independent of the choice of t_i due to the substitutivity properties of $\equiv$ and the maximal consistency of Φ.
10. the common case in many applications to other areas of mathematics as well as the ordinary mode of reasoning of informal mathematics in calculus and applications to physics, chemistry, engineering
11. according to De Morgan's laws

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External links

- Mortensen, Chris (2017). "Inconsistent Mathematics" (http://plato.stanford.edu/entries/mathematics-inconsistent/). *Stanford Encyclopedia of Philosophy*.